Conceptual completeness in categorical logic

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1 Theories as Categories

It is widely known that boolean algebras form the algebraic counterpart of propositional theories. It is less known to broader mathematical circles that certain classes of small categories (with particular properties) form the correct algebraic counterpart of classes of first-order theories (with particular syntactic complexity). This idea has a precise mathematical content. In the less obvious direction, given a (possibly many-sorted) first-order theory of the appropriate kind, one constructs a category by allowing the objects to be formulae in context $\varphi(\boldsymbol{x})$ and the morphisms from such an object to $\psi(\boldsymbol{y})$ to be equivalence classes, up to mutual provability within the theory, of provably (again within the theory) functional relations $\theta(\boldsymbol{x}, \boldsymbol{y})$ from the former to the latter, i.e such that $\theta \vdash_{\boldsymbol{xy}} \varphi \wedge \psi$. Being provably functional means that the sequents $\theta \wedge \theta(\boldsymbol{z}/\boldsymbol{y}) \vdash_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}} \boldsymbol{y} = \boldsymbol{z}$ and $\varphi \vdash_{\boldsymbol{x}} \exists \boldsymbol{y} \theta$ are provable. Composition of morphisms

$$\gamma(\boldsymbol{y}, \boldsymbol{z}) \cdot \theta(\boldsymbol{x}, \boldsymbol{y}) \colon \varphi(\boldsymbol{x}) \to \psi(\boldsymbol{y}) \to \chi(\boldsymbol{z})$$

is represented by the formula $\exists \boldsymbol{y}(\gamma \wedge \theta)$ [6]. The usual Lindenbaum algebra of a propositional theory is a special instance of this construction, viewing the boolean algebra as a partially ordered set, hence as a category.

We focus on regular and coherent theories. Regular theories consist of sequents $\varphi \vdash_{\boldsymbol{x}} \psi$, where φ, ψ are built from atomic formulae by $\wedge$ and $\exists$. Coherent theories allow further the use of $\vee$ in the formation of formulae. The latter have the same expressive power as full first-order logic, allowing appropriate modifications of language [4]. The algebraic counterpart of a regular theory is that of a regular category, i.e one with finite limits and regular epi - mono factorizations (sufficient for expressing $\exists$) that are stable under pullback ($\exists$ is compatible with substitution of terms). The counterpart of a coherent theory is that of a coherent category, i.e a regular category where finite suprema of subobjects exist.

Under the above correspondence, models of theories are just regular (respectively, coherent) functors to the category of sets, i.e functors that preserve the relevant constructions (finite limits and regular epis in the case of regular logic, as well as finite suprema, in the case of coherent logic). One advantage of the categorical approach is that models in any category, where the relevant constructions are available, make natural sense. Regular (coherent) functors $F \colon \mathcal{C} \to \mathcal{D}$ between small regular (coherent) categories are just interpretations of one theory within another.

2 Conceptual Completeness

The categorical perspective allows for a question that is not even possible to formulate within classical model theory. Notice that models of a theory are now organized as a category. Its objects are regular (coherent) functors $F: \mathcal{C} \to$ Sets and its morphisms are natural transformations between them which, under the above correspondence, amount to homomorphisms between models. Regular (coherent) categories with regular (respectively, coherent) functors are organized in a 2-category REG (respectively, COH).

An interpretation of theories $F: \mathcal{C} \to \mathcal{D}$ induces by restriction a functor between the respective categories of models

$$- \cdot F: \mathrm{REG}(\mathcal{D}, \mathrm{Sets}) \to \mathrm{REG}(\mathcal{C}, \mathrm{Sets}) \quad (*).$$

(Accordingly for COH.) The question that naturally arises is: If this functor is an equivalence of categories, what can be said about the interpretation F itself? Is it an equivalence of categories as well? The straight answer is no. But it induces an equivalence at the level of appropriate *completions* of the given categories. In both the regular and coherent case this completion process involves the *effectivization* $\mathcal{C}_{ef}$ of a regular category $\mathcal{C}$. This is the process of universally turning it into an effective (=Barr-exact) one, i.e making every equivalence relation the kernel pair of its coequalizer [7]. In the coherent case one adds first finite coproducts, so that existing ones are preserved (the *positivization* of the coherent category, [6]). The combined process yields the pretopos (a notion originating in A. Grothendieck's approach to Algebraic Geometry) associated with a coherent category. In either case, regular or coherent, the completed category of a syntactical category $\mathcal{C}_{\mathbb{T}}$ of a theory $\mathbb{T}$ is nothing else than the syntactical category of the theory $\mathbb{T}^{eq}$, introduced by S. Shelah (exactly by adding new sorts for quotients of definable equivalence relations, [5]).

For the coherent case, already since [9], we know that whenever the functor $(*)$ above is an equivalence, then F induces an equivalence at the level of associated pretoposes. The proof was model-theoretic, by an argument involving the compactness theorem and the method of diagrams. A. Pitts improved on that, giving a categorical proof, valid over any base topos with a natural number object when allowing models to take values in a *sufficient class of toposes* ([10], in particular Definition 2.3) and equivalence to mean a fully faithful functor which is essentially surjective on objects. His argument involved the topos of filters construction and the calculus of relations inside a topos. For regular logic a similar result holds: If the regular functor F between regular categories induces an equivalence $(*)$ between the respective categories of models in sets, then it induces an equivalence between the respective effectivizations. This is an immediate consequence of the main result in [8]. The argument is again model-theoretic and involves choice principles. A purely categorical, intuitionistically valid argument, exploiting the result of Pitts on pretoposes, is the main contribution (in logical terms) of [1].

Theorem 1. *(Conceptual Completeness for Regular Logic, intuitionistically) Let $F: \mathcal{C} \to \mathcal{D}$ be a regular functor such that, for all toposes $\mathcal{V}$ in a sufficient* *class, the induced functors between the categories of models*

$$- \cdot F\colon \mathrm{REG}(\mathcal{D}, \mathcal{V}) \to \mathrm{REG}(\mathcal{C}, \mathcal{V})$$

are equivalences. Then the induced $F_{ef} : \mathcal{C}_{ef} \to \mathcal{D}_{ef}$ *between effectivizations is an equivalence of categories.*

3 Applications

Regular theories arise frequently in mathematical practise. Their categorical renditions and the respective effectivizations may account for quite complicated constructions. When applied to (the effectivization of the category associated to) the regular theory of the representation of a graph (quiver) inside finitely presentable R-modules yields M. Nori's category of motives. The universal property of such a category can be subsequently cast in purely category-theoretic terms [2].

The improved intuitionistic version of conceptual completeness can also be of use. Theories internal in a topos are not all that exotic. For rings R, S inside a topos (sheaves of rings, in plain terms) that are internally coherent, the theories of flat modules are regular (internal) theories -notice here that the claim of [8], 6.1, that this is so for general rings is erratic. Equivalence of their (indexed) categories of flat modules yields an equivalence mod-$R \simeq$ mod-S of (internal categories of internally) finitely presentable modules. This might simplify rather complicated situations studied in Algebraic Geometry, as [3] shows.

References

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