UNIFORMITY FUNCTIONS IN DESCRIPTIVE SET THEORY AND THEIR APPLICATIONS

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(CONTAINS JOINT WORK WITH TAKAYUKI KIHARA AND KENG MENG NG)

1. Introduction

Suppose that X, Y are non-empty sets and that $P \subseteq X \times Y$ satisfies the property that for all $x \in X$ there is some $y \in Y$ such that $(x, y) \in P$. From the Axiom of Choice one can obtain a *choice-function* $u : X \to Y$ such that for all $x \in X$ we have $(x, u(x)) \in P$. We are concerned with the question of finding a "definable" choice-function. Here our underlying spaces are Polish, i.e., the topological spaces that arise from complete separable metric spaces. We give two examples of choice-functions, in the first the choice-function is Borel-measurable, and in the second it is continuous. Both these examples have important consequences in seemingly unrelated topics. The former is joint work with Takayuki Kihara and Keng Meng Ng.

We begin with some comments on notation. Given a set P we write $P(x)$ instead of $x \in P$. If $P \subseteq X \times Y$ and $x \in X$ we denote by P_x the x-section $\{y \in Y \mid P(x, y)\}$ of P. We identify the natural numbers with the first infinite ordinal ω. The family of the *Borel* subsets of $\mathcal{X}$ is the least family that contains all open subsets of $\mathcal{X}$ and is closed under the operations of the countable union and the complement. We recall the finite levels of the usual hierarchy of the Borel sets in Polish spaces,

$$\begin{aligned} \utilde{\mathbf{\Sigma}}^0_1 &= \text{all open sets}, \\ \utilde{\mathbf{\Sigma}}^0_{n+1} &= \{\cup_{i \in \omega} A_i \mid \text{all } A_i \text{ are subsets of some Polish space } \mathcal{X} \text{ and } \mathcal{X} \setminus A_i \in \utilde{\mathbf{\Sigma}}^0_n\}, \\ \utilde{\mathbf{\Pi}}^0_n &= \{B \subseteq \mathcal{X} \mid \mathcal{X} \setminus B \in \utilde{\mathbf{\Sigma}}^0_n\}, \end{aligned}$$

where $n \geq 1$. For example $\utilde{\mathbf{\Pi}}^0_1$ is the class of all open sets, $\utilde{\mathbf{\Sigma}}^0_2$ the class of all F_σ sets and so on. Given a class Γ and a Polish space $\mathcal{X}$ we denote by $\Gamma \upharpoonright \mathcal{X}$ the family of all subsets of $\mathcal{X}$ that belongs to Γ.

We also recall the *Baire space* $\mathcal{N} = \omega^\omega$, i.e., the Polish space of all sequences of natural numbers with the product topology. The members of the Baire space are denoted by lowercase Greek letters, α, β, γ, $\ldots$. A subset A of a Polish space $\mathcal{X}$ is *analytic* if it is the projection along the Baire space of a closed set $F \subseteq \mathcal{X} \times \mathcal{N}$.

A set $G \subseteq \mathcal{N} \times \mathcal{X}$ *parametrizes* $\Gamma \upharpoonright \mathcal{X}$, where Γ is a class of sets, if for all $P \subseteq \mathcal{X}$ we have that $P \in \Gamma$ exactly when $P = G_\alpha$ = the α-th section of G for some $\alpha \in \mathcal{N}$. We call the preceding α a *code for* P. If G parametrizes $\Gamma \upharpoonright \mathcal{X}$ and G is also a member of Γ, we say that G is *universal* for $\Gamma \upharpoonright \mathcal{X}$. We say that a class Γ is *parametrized* if for every Polish space $\mathcal{X}$ the family $\Gamma \upharpoonright \mathcal{X}$ is parametrized.

The preceding classes $\utilde{\mathbf{\Sigma}}^0_n$ are all parametrized by universal sets $(\mathrm{G}^{\mathcal{X}}_n)_{\{\mathcal{X}\colon \text{Polish}\}}$ in a natural way, for example $\mathrm{G}^{\mathcal{X}}_1$ consists of all pairs (α, x) such that $x \in \cup_{i \in \omega} V(\mathcal{X}, \alpha(i))$,

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where $(V(\mathcal{X}, s))_{s\in\omega}$ is a fixed enumeration of the basis of $\mathcal{X}$. Also the class of analytic sets is parametrized by universal sets. The class of Borel sets is parametrized as well, although the parametrizing sets cannot be universal. Thus we can talk about *Borel codes*, *analytic codes* and $\utilde{\mathbf{\Sigma}}^0_n$ *codes*. In fact in the latter two cases every $\alpha \in \mathcal{N}$ is an analytic (respectively $\utilde{\mathbf{\Sigma}}^0_n$) code for some possibly empty subset of the given Polish space $\mathcal{X}$.

2. The example of a Borel-measurable choice-function

In many naturally occurring cases of Borel sets $P \subseteq \mathcal{X} \times \mathcal{Y}$ it is possible to find Borel-measurable choice-functions. These include the cases where for all $x \in \mathcal{X}$ we have that: P_x is (a) compact, or (b) countable, or (c) non-meager, or (d) of positive measure (for some fixed σ-finite Borel-measure). The area of *effective descriptive set theory* explains the underlying cause for the validity of these results. This boils down to the fact that we can find points in the sections P_x, which are "definable from x". More specifically there is a canonical assignment $\tilde{d} : \mathcal{X} \to \mathcal{X}^\omega$ such that for each x, $\tilde{d}(x)$ is a sequence that enumerates all points that are "definable from x" (the $\Delta^1_1(x)$-points). Moreover this assignment is done in a Borel-way, so that if P is Borel and for all $x \in \mathcal{X}$, $P_x \cap \tilde{d}(x) \neq \emptyset$ then there is a Borel-measurable choice-function for P, namely $u(x) = \tilde{d}(x)(n_x)$, where n_x is the least index of a term in $\tilde{d}(x)$ that belongs to P_x. This is the Strong Δ-Selection Principle, see [10, 4D.6]. If P_x satisfies one of the preceding $(a) - (d)$ then $P_x \cap \tilde{d}(x) \neq \emptyset$. (This is a consequence of well-known theorems in effective descriptive set theory.) So from the preceding P admits a Borel-measurable choice-function.

This method has recently found an application in the theory of functions in Polish spaces [2]. To be more specific, suppose that $f : \mathcal{X} \to \mathcal{Y}$ is $\utilde{\mathbf{\Sigma}}^0_n$-measurable; the latter means that $f^{-1}[U] \in \utilde{\mathbf{\Sigma}}^0_n \upharpoonright \mathcal{X}$ for all open $U \subseteq \mathcal{X}$. For the natural choice of universal sets $\mathrm{G}^{\mathcal{Z}}_m$ there is a continuous function u such that $u(\alpha)$ is a $\utilde{\mathbf{\Sigma}}^0_n$-code for the open set that is encoded by α, i.e., $\mathrm{G}^{\mathcal{Y}}_1(\alpha, f(x)) \iff \mathrm{G}^{\mathcal{X}}_n(u(\alpha), x)$ for all $x \in \mathcal{X}$; in this case we say that $f^{-1}\utilde{\mathbf{\Sigma}}^0_1 \subseteq \utilde{\mathbf{\Sigma}}^0_n$ is satisfied with a *continuous function in the codes*.

Now let us consider the case where $f : \mathcal{X} \to \mathcal{Y}$ is $\utilde{\mathbf{\Pi}}^0_n$-measurable for some $n \geq 2$. The latter means that $f^{-1}[U] \in \utilde{\mathbf{\Pi}}^0_n$ for all open $U \subseteq \mathcal{Y}$. We denote this by $f^{-1}\utilde{\mathbf{\Sigma}}^0_1 \subseteq \utilde{\mathbf{\Pi}}^0_n$. This is a somewhat unusual property, and the only examples of such functions that we are aware of are the functions for which there is a sequence $(X_i)_{i\in\omega}$ of $\utilde{\mathbf{\Pi}}^0_{n-1}$ sets such that $\mathcal{X} = \cup_i X_i$ and each restriction $f \upharpoonright X_i$ is a $\utilde{\mathbf{\Sigma}}^0_{n-1}$-measurable function. Whether these are the only functions that are $\utilde{\mathbf{\Pi}}^0_n$-measurable is an important open question that remains open for $n \geq 3$. The answer is affirmative for $n = 2$ (Jayne-Rogers Theorem [3]) and it is conjectured that it is also affirmative for all $n \geq 2$. In fact there is an even stronger conjecture:

Conjecture 2.1. *Suppose that $\mathcal{X}$, $\mathcal{Y}$ are Polish spaces and that $f : \mathcal{X} \to \mathcal{Y}$ is a function with the property that $f^{-1}\utilde{\mathbf{\Sigma}}^0_m \subseteq \utilde{\mathbf{\Sigma}}^0_n$, where n, m are fixed naturals with $2 \leq m \leq n$. Then there is a sequence $(X_i)_{i\in\omega}$ of $\utilde{\mathbf{\Pi}}^0_{n-1}$ sets such that $\mathcal{X} = \cup_i X_i$ and each restriction $f \upharpoonright X_i$ is a $\utilde{\mathbf{\Sigma}}^0_{n-m+1}$-measurable function.*

Kihara [5] showed that for all (n, m) with $2 \leq n < 2m - 1$ the functions that satisfy the preceding conjecture at (n, m) are exactly the ones that satisfy the condition $f^{-1}\utilde{\mathbf{\Sigma}}^0_m \subseteq \utilde{\mathbf{\Sigma}}^0_n$ with a continuous function in the codes. Unlike the case of $\utilde{\mathbf{\Sigma}}^0_n$-measurability we do not know if the latter is always true, i.e., if the condition

$f^{-1}\utilde{\mathbf{\Sigma}}^0_m \subseteq \utilde{\mathbf{\Sigma}}^0_n$ is always satisfied with a continuous function in the codes. However, a major tool from effective descriptive set theory, namely the *Louveau Separation* [9] establishes the existence of α-definabe points in the pre-image of $(\mathrm{G}^{\mathcal{Y}}_m)_\alpha$ under f. In other words we have $f^{-1}[(\mathrm{G}^{\mathcal{Y}}_m)_\alpha] \cap \tilde{d}(\alpha) \neq \emptyset$. So from the preceding we obtain

Theorem 2.2 (G.-Kihara-Ng, see [2])**.** *Suppose that $\mathcal{X}$, $\mathcal{Y}$ are Polish spaces and that $f : \mathcal{X} \to \mathcal{Y}$ satisfies $f^{-1}\utilde{\mathbf{\Sigma}}^0_m \subseteq \utilde{\mathbf{\Sigma}}^0_n$ for some $m, n \geq 1$. Then the condition $f^{-1}\utilde{\mathbf{\Sigma}}^0_m \subseteq \utilde{\mathbf{\Sigma}}^0_n$ is satisfied with a Borel-measurable function in the codes.*

Using the preceding result together with some deep results from recursion theory we make progress on the conjecture above. More specifically it is proved in [2] that Conjecture 2.1 is reduced to the instances $(2, n)$, i.e., if the conjecture is true for all $n \geq 2 = m$ then it is also true for all $n \geq m \geq 2$.

3. The example of a continuous choice-function

Unlike the preceding method, where the choice-functions are established by the existence of definable points, *continuous* choice-functions are usually obtained by a direct construction. Here we offer the example of one such choice-function that is related to the notion of convexity. A subset A of $\mathbb{R}^N$, where $N \geq 1$, is *convexly generated* if it belongs to the least family, which contains all compact convex sets (equivalently all open convex sets) and is closed under countable increasing unions as well as countable intersections. The family of convexly generated subsets of $\mathbb{R}^N$ is parametrized and therefore we can talk about *convexly generated codes.*

It is evident that all convexly generated subsets of $\mathbb{R}^N$ are Borel. Klee [6] asked whether the converse is correct, and has answered this affirmatively in the case $N = 2$, see [7]. Larman [8] proved the analogous result in the case $N = 3$, and finally Preiss showed that the answer is affirmative for all $N \geq 1$. In fact Preiss showed the following stronger assertion that is known as *Preiss Separation* see [11] and also [4, 28.15]: Suppose that A, B are disjoint analytic subsets of $\mathbb{R}^N$; if A is convex then there is a Borel set C that *separates A from B*, i.e., $A \subseteq C$ and $C \cap B = \emptyset$. The proof of the latter result is by contradiction. We are able to give a constructive proof, and as a consequence we obtain a continuous choice-function in the codes.

Theorem 3.1 (G. see [1])**.** *Suppose that $N \geq 1$. Then there is a continuous* (*in fact recursive*) *function $u : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ such that for all $\alpha, \beta \in \mathcal{N}$, if α is an analytic code for a convex set A and β is an analytic code for a set B that is disjoint from A, then $u(\alpha, \beta)$ is a convexly generated code for a Borel separating set C.*

A notable consequence to the preceding result is

Corollary 3.2. *For all $A \subseteq \mathbb{R}^N$, $N \geq 1$, the following are equivalent:*

- (i) *A is convex and has a recursive Borel code;*
- (ii) *A admits a recursive convexly generated code.*

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