ON MATHEMATICAL APPLICATIONS OF LARGE CARDINAL AXIOMS

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Abstract. The various large cardinal axioms have been intensively studied during the last decades and have proven to be a very important and fruitful set-theoretic theme, with several mathematical applications. In this survey talk, our aim is to give a brief overview of the large cardinal hierarchy, while pointing at some indicative examples of notions whose reflective nature has turned out to be useful in other mathematical contexts.

1. Extended abstract

The first-order axiomatic system of ZFC, that is, of Zermelo-Fraenkel set theory with the Axiom of Choice, is an axiomatic framework that can encompass the vast majority of the entire mathematical edifice, thus serving as the (widely accepted) current foundation of mathematics.

However, and as a consequence of some groundbreaking mathematical advances of the previous century (with central examples being Gödel's results and Cohen's introduction of the method of *forcing*), there exists an extensive and ever-expanding list of mathematical statements, from diverse areas of mathematics, which are provably *independent*; that is, statements that can be neither proved nor disproved from the ZFC axioms. In other words, by now, it is common knowledge that ZFC set theory is unable to settle many important mathematical problems, among which there are prominent examples such as Cantor's *Continuum Hypothesis*. Consequently, it is natural to wonder whether there are additional axioms that, when added to ZFC, would result in a stronger theory that is able to resolve (at least some of) these independent problems.

1.1. **Large cardinals.** One important family of candidates for new axioms consists of the so-called *large cardinal axioms*. Roughly, such postulates, which were first considered in the early 20th century, assert the existence of certain strong forms of infinity (that is, of infinite sets satisfying certain strong properties), ones that are not deducible from ZFC alone. To give an example, an uncountable cardinal κ is called *weakly inaccessible* if it is both regular and a limit cardinal.[1] We note that the definition of weak inaccessibility generalizes, in the realm of uncountable cardinals, properties that are already satisfied by $\omega = \aleph_0$. The existence of a weakly inaccessible cardinal implies the existence of a set model for ZFC and, hence, it cannot be proved in ZFC, unless ZFC is inconsistent.

The notion of weak inaccessibility is actually one of the "weakest" large cardinal notions that have been considered; indeed, there are many "stronger" notions, among which we find *weakly compact*, *Ramsey*, *measurable*, *strong*, *Woodin*, *strongly*

[1] Recall that an infinite cardinal κ is called *regular* if it cannot be written as the union of less than κ many sets, each of size less than κ. Under the Axiom of Choice, every successor cardinal, such as $\aleph_1$, $\aleph_{59}$ and $\aleph_{\omega_1+31}$, is regular. An infinite cardinal is called *singular* if it is not regular; it follows that every singular cardinal is necessarily a limit cardinal. The first example of a singular cardinal is that of $\aleph_\omega$, which can be written as a countable union of smaller sets, namely: $\aleph_\omega = \bigcup_{n\in\omega} \aleph_n$.

compact, *supercompact*, *extendible* cardinals, and others. In fact, the list of large cardinals has grown considerably over the years, having been enriched with notions coming from a wide spectrum of mathematical interests. Yet, it is an impressive fact that these postulates are found to be linearly ordered in consistency strength, forming an increasing hierarchy of stronger and stronger axioms of infinity (where "stronger" — and, respectively, "weaker" — refers to consistency strength). Using this hierarchy, we are then able to "measure", via comparison, the consistency strength of any independent set-theoretic (and, thus, mathematical) statement.

1.2. **Reflection, compactness, and applications.** An essential characteristic of many large cardinal notions is their inherently *reflective* nature. This is an important and general feature that makes these notions both amenable to various techniques and, also, widely applicable to several families of mathematical problems. Intuitively, and in broad terms, reflection can be described by saying that, if a given structure satisfies some particular property, then there must already exist some "small substructure" of it that satisfies the same property. In other words, the property at hand must already "reflect" to something "smaller" where, typically, "smaller" means smaller in size. Dually, the concept of *compactness* can be described by saying that, if some property holds for every small substructure of a given structure, then the same property holds for the structure itself. Note that these concepts provide us with a very general framework, able to capture structures and properties from practically any mathematical area. Indeed, many problems in mathematics can ultimately be rephrased as questions about reflection or, dually, about compactness.

It is exactly in this respect that various large cardinals unfold their strength and, thus, have significant import and influence on such issues. In fact, several well-known large cardinal notions have traditional characterizations that are of such flavor. For example, an uncountable cardinal κ is *weakly compact* if and only if whenever a complete graph of size κ is edge-colored, using two colors, there is a complete subgraph of size κ that is monochromatic, i.e., whose edges take all the same color.[2] Moving to even stronger notions, such as ω_1-*strongly compact* and *strongly compact* cardinals, and as an indication, let us give some specific results that highlight the aforementioned influence, providing concrete applications in other mathematical fields.

Theorem 1 (Bagaria & Magidor [4]). *If κ is ω_1-strongly compact, then:*

(1) *Every first-countable non-metrizable topological space X has a non-metrizable subspace $Y \subseteq X$ such that $|Y| < \kappa$.*
(2) *Every graph G with chromatic number $\chi(G) > \aleph_0$ has a subgraph $H \subseteq G$ with $|H| < \kappa$ and $\chi(H) > \aleph_0$.*

Theorem 2 (Bagaria & Magidor [4]). *The cardinal κ is ω_1-strongly compact if and only if every product of Lindelöf spaces is κ-Lindelöf.*

Theorem 3 (Magidor [6]). *Suppose that κ is a strongly compact cardinal and that X is a first-countable topological space. If every subspace of X of cardinality less than κ is metric, then X itself is also metric.*

Results involving ω_1-strongly compact cardinals have also emerged in the field of Algebra and, specifically, in Group Theory; see [3]. For even further applications of (other) large cardinals in Category and Homotopy Theory see [2] and [5].

Towards a general framework, let φ be any property of some relevant family of mathematical structures (e.g., topological spaces, metric spaces, groups, etc.),

[2]Note that this, in effect, is an uncountable generalization of Ramsey's theorem. Weakly compact cardinals have many other (equivalent) reformulations, of similar flavor.

expressible over any logic (e.g., first-order, second-order, etc.). We assume that φ is a property that is preserved under isomorphisms of the relevant structures, which is a very natural assumption. It should be noted that this is already an extremely general setting, where the vast majority of mathematical properties can be defined and studied. Then, we make the following central definition:

Definition 4 (Magidor [6]). A cardinal κ is called a **reflection cardinal for** φ if, given any relevant structure $\mathcal{A}$ such that $\varphi(\mathcal{A})$ holds, there is a substructure $\mathcal{B} \subseteq \mathcal{A}$ with $|\mathcal{B}| < \kappa$ such that $\varphi(\mathcal{B})$ holds.

Depending on the particular complexity of the (expressible) property φ, we are able to gauge the strength of the large cardinal κ that serves as a reflection cardinal for this property. In other words, given a mathematical property φ of some family of structures of interest, we can effectively find the large cardinal assumption that reflects the property. Some basic examples of this correlation are:

Theorem 5 (Magidor [6]). *If κ is supercompact, then κ is a reflection cardinal for every property of structures that is Σ_2-expressible (in the language of set theory).*

Theorem 6 (Magidor [6]). *If κ is extendible, then κ is a reflection cardinal for every property of structures that is Σ_3-expressible (in the language of set theory).*

As the complexity of (the defining formula of) the property of interest grows, we need the assumption of even stronger principles. In fact, by work of Bagaria, we have an exact level-by-level correspondence between the complexity of properties and the relevant large cardinals notions; these are the so-called $C^{(n)}$-extendible cardinals — see [1].

1.3. **This talk.** In this survey talk, we will start by giving a brief overview of the hierarchy of large cardinal axioms. Subsequently, we will concentrate on some specific notions and their properties, presenting in some more detail how their reflective nature has turned out to be useful in other mathematical contexts, such as the ones mentioned above.

References

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